1914
W699

WILSON

The Calculus for Functions Defined over a Perfect Set

Mathematics

A. M.

1914

UNIV. OF
ILLINOIS
LIBRARY

THE UNIVERSITY
OF ILLINOIS
LIBRARY
1914
W699

THE CALCULUS FOR FUNCTIONS DEFINED OVER A PERFECT SET

BY

WILLIAM HAROLD WILSON

A.B. Albion College, 1913

THESIS

Submitted in Partial Fulfillment of the Requirements for the

Degree of

MASTER OF ARTS

IN MATHEMATICS

IN

THE GRADUATE SCHOOL

OF THE

UNIVERSITY OF ILLINOIS

1914

1914
W699

UNIVERSITY OF ILLINOIS

THE GRADUATE SCHOOL

May 28 1914

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

William Harold Wilson

ENTITLED THE CALCULUS FOR FUNCTIONS DEFINED OVER A PERFECT SET.

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE OF Master of Arts.

Arthur R. Crathorne
In Charge of Major Work

J. Townsend
Head of Department

Recommendation concurred in:

Committee on Final Examination

284668

1914
W699

PREFACE.

***

The object of this thesis is to investigate the fundamental theorems of the Calculus for functions defined over a perfect set. The ordinary text in Calculus deals only with functions that are defined over the continuum. In Chapter I the perfect set is defined and discussed in comparison with the continuum.

Chapter II deals with the definition of a function, the definitions and laws of operation connected with limits and the definition of continuity, giving in particular some theorems concerning the continuity of certain functions defined over a perfect set. The four fundamental limits are discussed and their general properties are expressed as corollaries of the general theorems. The succession of topics in this chapter closely follows the traditional arrangement of the ordinary text-book on Calculus.

A discussion of derivatives, the law of the mean, and Rolle's Theorem is the most important part of Chapter III. The determination of the character of a function by means of its derivatives and also the relationship of derivatives of functions defined over a perfect set and over the continuum are briefly discussed. The general outline of the text-books was not followed here as the work which was developed seems to show far more possibilities of development than the usual work of the texts.

Digitized by the Internet Archive
in 2013

http://archive.org/details/calculusforfunct00wils

TABLE OF CONTENTS.

♀ ♀ ♀ ♀

Chapter I.

SETS OF POINTS, THE PERFECT SET AND ITS PROPERTIES.

I	Sets	Page 1
II	Elementary Definitions	2
III	Properties and Classes of Sets of Points	4
IV	Perfect Sets	6
V	The Perfect Set and the Continuum	10
VI	Theorems Relating to Perfect Sets	10

Chapter II.

FUNCTIONS AND THE THEORY OF LIMITS.

I	Constants and Variables	Page 16
II	Definition of a Function	16
III	Definition of a Limit	18
IV	Continuity of Functions	21
V	Laws of Operations with Limits	25
VI	Continuity of Special Functions	36

Chapter III.

THE DERIVATIVE AND SOME OF ITS PROPERTIES.

I	Definition of a Derivative	Page 42
II	The Law of the Mean	44
III	Determination of a Function by Means of Derivatives	46
IV	Relationship of Derivatives	48

NOTATION USED.

******

$\underset{x=a}{\mathfrak{L}}\, f(x)$ = the limit of the function $f(x)$ at the point a.

$\underset{x=a}{R\mathfrak{L}}\, f(x)$ = the right-hand limit of the function $f(x)$ at the point a.

$\underset{x=a}{L\mathfrak{L}}\, f(x)$ = the left-hand limit of the function $f(x)$ at the point a.

$\underset{x=a}{U\mathfrak{L}}\, f(x)$ = the unilateral (right or left) limit of the function $f(x)$ at a.

$Y(\delta)$ = the upper limit of a function in the interval $(a, a + \delta)$.

$\Lambda(\delta)$ = the lower limit of a function in the interval $(a, a + \delta)$.

$\underset{x=a}{R^{u}\mathfrak{L}}\, f(x)$ = the upper right-hand limit of the function $f(x)$ at $x = a$.

$\underset{x=a}{R_{1}\mathfrak{L}}\, f(x)$ = the lower right-hand limit of the function $f(x)$ at $x = a$.

$\underset{x=a}{L^{u}\mathfrak{L}}\, f(x)$ = the upper left-hand limit of the function $f(x)$ at $x = a$.

$\underset{x=a}{L_{1}\mathfrak{L}}\, f(x)$ = the lower left-hand limit of the function $f(x)$ at $x = a$.

$D^{r}f(x)$ = the upper right-hand derived number of $f(x)$ over the continuum.

$D_{r}f(x)$ = the lower right-hand derived number of $f(x)$ over the continuum.

$D^{l}f(x)$ = the upper left-hand derived number of $f(x)$ over the continuum.

$D_{l}f(x)$ = the lower left-hand derived number of $f(x)$ over the continuum.

$D_{(P)}f(x)$ = the derivative of the function $f(x)$ over the set P.

CHAPTER I

***

SETS OF POINTS, THE PERFECT SET AND ITS PROPERTIES.

***

"The theory of linear sets of points arose historically from the discussion of questions connected with Fourier's series and of the functions which can be represented by such series."*

1. SETS. The objects dealt with in Mathematics may be points, lines, curves, positive integers, complex quantities, matrices, displacements, forces, velocities, men, animals and indeed anything which may lend itself to Mathematical considerations. Such objects are individuals; they may be infinite in number but they are still individuals. Moreover, they may present themselves in sets.° These sets or, as they are sometimes called, classes or aggregates may consist of a finite or an infinite number of objects or elements. Examples of such sets are:**

1) All rational numbers.

2) All lines through a given point in space.

3) All planes of symmetry of a given regular polyhedron.

4) All substitutions which can be performed on any finite number of letters.

5) All translations of a plane with respect to a set of perpendicular

---

*E. W. Hobson, *The Theory of Functions of a Real Variable*, p. 57. See also, Edward B. Van Vleck, *The Influence of Fourier's series upon the Development of Mathematics*, Science, vol. 39, (1914), p. 120.

°Maxime Bocher, *The Fundamental Conceptions and Methods of Mathematics*, Bulletin of the American Math. Soc., Series 2, vol. 11, p. 115.

**Maxime Bocher, *Introduction to Higher Algebra*, p. 80.

axes within it.

6) All integers between *1* and *100*.

7) All letters of the alphabet.

Thus it will be seen that the conception of a set is exceedingly general.

It should be noted here that in most of the cases with which one deals in Mathematics, there are one or more relations which exist between the elements. A large number of relations may be involved. Some of these relations may connect two objects, as A is greater than B, some may connect three objects, as $A + B = C$, etc. These relations theoretically suffice to determine whether a new object does or does not belong to the set in consideration. Thus if we consider inequality as a relation, then the positive integers form a set,

$$1, 2, 3, \cdots\cdots\cdots, n, \cdots$$

even though all other relations may be excluded from consideration.

DEFINITION. *Such a set, with its attendant relations, is called a mathematical system, or simply a system.*

In general, the order in which the objects are taken is by no means immaterial. Examples of this are found in vector products, and in the multiplication of matrices where in general

$$ac \neq ca.$$

2. ELEMENTARY DEFINITIONS. In the study of Calculus, sets of numbers are considered. Each number of a given set or, as it is sometimes called, assemblage may be represented by a single point on a straight line; thus to an aggregate of numbers, there corresponds an aggregate of points on a straight line.* The order of the number continuum is the same as

*E. W. Hobson, loc. cit., p. 57.

that of the point continuum which corresponds to it; if B lies between A and C in the number continuum, then B lies between A and C in the point continuum. For this reason we speak of sets of numbers and sets of points indiscriminately, using the expression, sets or aggregates of points in preference, however, since it also conveys the geometrical ideas involved. We will here call such sets *point aggregates.**

Some definitions concerning point aggregates follow:

DEFINITION I. *Two point aggregates A and B are equal, $A = B$, when every point of A lies in B and every point of B lies in A.*

DEFINITION II. *If every point of an aggregate B lies in an aggregate A but not every point of A lies in the aggregate B, then we say that A is greater than B or that B is less than A, and write*

$$A > B \text{ or } B < A.$$

DEFINITION III. *If A does not exist, that is, it contains no points, then we say that A is equal to zero and write*

$$A = 0.$$

DEFINITION IV. *The fact that A contains at least one point we denote by saying that A is greater than zero,*

$$A > 0.$$

DEFINITION V. *Let A and B be two point aggregates that have no point in common. The aggregate formed by their reunion is called their sum and is denoted by*

$$A + B.$$

DEFINITION VI. *If the aggregate B is less than the aggregate A, the aggregate formed by removing all the points of B from A is called the dif-*

---

*This term is used by Pierpont, *The Theory of Functions of Real Variables.*

ference of A, B, and is denoted by

$$A - B.$$

This is called the complement of B.

3. Properties and Classes of Sets of Points.

Definition I. *An open interval* (α,β) *is defined by the totality of real numbers which are greater than* α *and less than* β. *A closed interval is defined by the totality of real numbers which are greater than or equal to* α *and less than or equal to* β.

It is seen from this definition that the only difference between the open interval (α,β) and the closed interval (α,β) is that the end-points α and β are excluded from the former and included in the latter.

As a result of this definition we also have

Definition II. *If in an interval* (α,β), *a smaller one* (α',β') *such that* α *is equal to or less than* α' *and* β *is equal to or greater than* β' *is taken, then the latter interval is called a sub-interval of the former.*

Definition III. *An interval* (α,β), *arbitrarily small, is called the neighborhood of a point x lying within it.*

The neighborhood of a point x is often denoted by $(x - \varepsilon, x + \varepsilon)$, where ε is arbitrarily small. If the interval does not include the point, as is the case with respect to the neighborhood of 0 in the following set,

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\ldots\ldots\ldots, \frac{1}{n},$$

then the neighborhood is called a deleted neighborhood.

Definition IV. *Any point is a limiting point of a set if in the neighborhood of the point there is an infinity of points of the set.*

An illustration of this is found in the set above in which 0 is a

limiting point. Another example is the following set in which 0 is again a limiting point:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots, \frac{1}{n}, \ldots\ldots.$$

A limiting point is not necessarily a point of the set as is shown by the examples just given. The following, however, is an example of a set in which the limiting point is included:

$$1, \frac{1}{2}, \frac{1}{2}+\frac{1}{3}, \frac{1}{2}+\frac{1}{4}, \frac{1}{2}+\frac{1}{5}, \ldots\ldots\ldots, \frac{1}{2}+\frac{1}{n}, \ldots.$$

The limiting point in this case is $\frac{1}{2}$.

DEFINITION V. *Every point of a set which is not a limiting point is an isolated point; it can be enclosed in an interval so small that the interval will contain no other point of the set.*

Thus we see by definition that every point of a set is either a limiting point or an isolated point of the set.

DEFINITION VI. *If a set A contains no isolated points, it is dense-in-itself.**

DEFINITION VII. *If a set A contains all of its limiting points, it is said to be complete or closed.*

DEFINITION VIII. *If in an interval (α,β) in which a set of points A is contained, no sub-interval, however small can be found which does not contain points of A, then the set A is said to be everywhere-dense within the given interval.*

It is readily seen that a set which is everywhere-dense is also dense-

---

*Authors disagree on the term used here. This is the term used by E. W. Hobson, *The Theory of functions of a Real Variable.* Pierpont uses the simple term dense.

in-itself but the converse is not necessarily true.

DEFINITION IX. *The totality of limiting points of a set A constitute a set A' which is called a derived set of A or the first derivative of A.*

DEFINITION X. *If a set has a finite number of derived sets, it is said to belong to the first species. If a set has an infinite number of derivatives, it is of the second species.*

4. PERFECT SETS. From the definitions that have been given, we may define a perfect set at once. The definition is as follows:

DEFINITION. *If a set A is dense-in-itself and also complete, it is a perfect set.*

From the definition, it is obvious that a perfect set contains all of its limiting points and that every point of the set is a limiting point. Jordan applies the term perfect to sets which are merely complete.* Examples of perfect sets are the following:

EXAMPLE I. The set of points constituting the interval (α,β) is perfect. This is merely a segment of the continuum which is a perfect assemblage of points.

EXAMPLE II. Let us take an interval of length l. In the middle of this interval lay off another interval of length l_1 given by

$$l_1 = [a - \frac{a}{1+3}]l, \qquad 0 < a < 1.$$

Drop the inner of the three intervals so formed and retain the end-points as points of the set which we are to form. In each of the two remaining intervals lay off sub-intervals of length l_2 given by

---

*Jordan, *Cours d'Analyse*, vol. I, p. 19.

$$l_1 + l_2 = [a - \frac{a}{2 + 3}]l.$$

In each case drop the interval which lies in the middle of the original sub-interval; that is, the interval for which the length has been determined but retain the end-points as points of the set. Continue thus indefinitely according to the law

$$l_1 + l_2 + l_3 + \cdots\cdots + l_n = [a - \frac{a}{n + 3}]l$$

$$= al\,\frac{n + 2}{n + 3}.$$

For the case in which $a = \frac{2}{3}$, we have

$$l_1 = \frac{l}{2}.$$

Also for the sum of the first $n - 1$ terms,

$$\Sigma_{n-1} = \frac{2(n + 1)}{3(n + 2)}\,l.$$

Then

$$l\,\frac{2(n + 1)}{3(n + 2)} + l_n = l\,\frac{2(n + 2)}{3(n + 3)}$$

from which

$$l_n = \frac{2}{3(n + 2)(n + 3)}\,l.$$

We find the following values (*Fig. 1*):

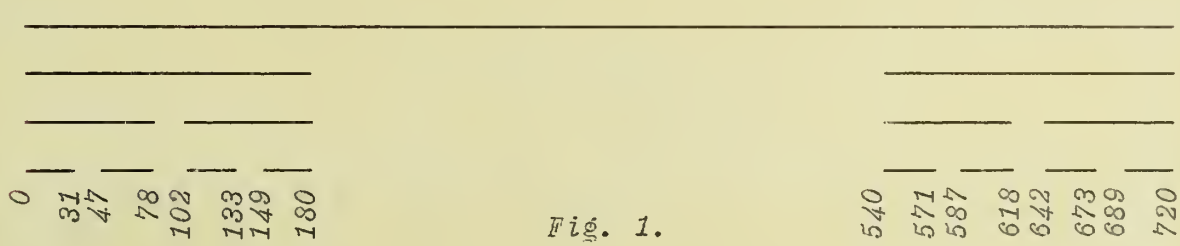

Fig. 1.

$$l_2 = \frac{l}{30}$$

$$l_3 = \frac{l}{45}$$

$$l_4 = \frac{l}{63}$$

$$l_5 = \frac{l}{84}$$

. . . .

When $l = 720$, we have the following points of the set:

$0, \cdot\cdot\cdot\cdot\cdot, 31, 47, \cdot\cdot\cdot\cdot\cdot, 78, 102, \cdot\cdot\cdot\cdot\cdot, 133, 149, \cdot\cdot\cdot\cdot\cdot, 180, 540, \cdot\cdot\cdot\cdot\cdot, 571, 587, \cdot\cdot\cdot\cdot\cdot, 618, 642, \cdot\cdot\cdot\cdot\cdot, 673, 689, \cdot\cdot\cdot\cdot\cdot, 720.$

Since each end-point of a free space is a limiting point and since all of these end-points are included in the set, it follows that the set is perfect. This set, however, is not everywhere-dense for there are intervals which contain no points of the set whatever.

It is interesting to note that in the formation of this set, each interval dropped is an integral part of the original interval; however, this would not be true for l different from 720 for then the parts would become fractional, as is the case here when we take all intervals after the one of length l_5. It is also interesting to note that for any value of a, we have

$$l_n = \frac{al}{(n+2)(n+3)}.$$

EXAMPLE III. Begin with the number 2 and take all intervals at the right of 2 which begin with an even number and end with the next odd number, (*Fig. 2*). Since the odd numbers are limiting points, they must be

included as part of the intervals. This set will appear with the follow-

Fig. 2.

ing points:

2, · · · · ·, 3, 4, · · · · ·, 5, 6, · · · · ·, 7, 8, · · · ·

EXAMPLE IV. Take an interval (α,β) which is closed and divide it into three equal parts, (*Fig. 3*). Drop the middle part, retaining the end-points. Divide each of the remaining intervals into three equal parts and drop the middle part in each case, always retaining the end-points. Continue this process indefinitely. Each end-point is a limiting point and

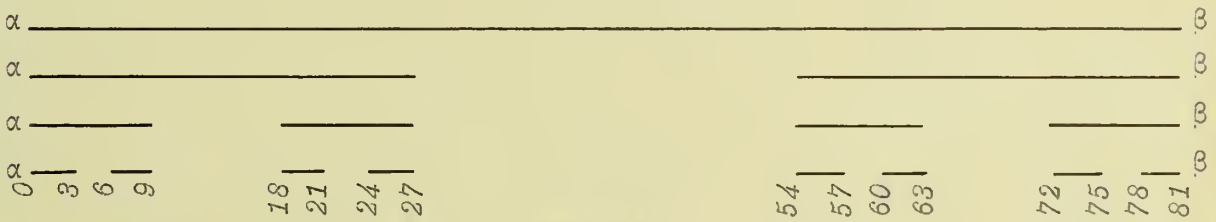

Fig. 3.

there are no other limiting points. Therefore the set of end-points is perfect. If we denote the length of the interval (α,β) by l and let $l = 81$, then the following points belong to the given set:

0, · · · · ·, 1, 2, · · · · ·, 3, 6, · · · · ·, 7, 8, · · · · ·, 9, 18, · · · · ·, 19, 20, · · · · ·, 21, 24, · · · · ·, 25, 26, · · · · ·, 27, 54, · · · · ·, 55, 56, · · · · ·, 57, 60, · · · · ·, 61, 62, · · · · ·, 63, 72, · · · · ·, 73, 74, · · · · ·, 75, 78, · · · · ·, 79, 80, · · · · ·, 81.

5. The Perfect Set and the Continuum. In the continuum formed of all real numbers, it is readily seen that every point is a limiting point and that all points are included; thus the continuum is a perfect set. It is therefore necessary that every property deduced for the perfect set must also hold true for the continuum as a special case of a perfect set. The most general case is evidently that in which there are intervals in which the points of the set are everywhere-dense and also intervals which contain no points of the set.

6. Theorems Relating to Perfect Sets.

Theorem I. *If an assemblage is perfect, then its derived assemblages are equal to each other and to the original assemblage, that is,*

$$M = M' = M'' = \cdot \cdot \cdot \cdot \cdot \cdot \cdot .$$

By definition, the first derived set contains every limiting point of the original set; also it contains no other point. But since every point of a perfect set is a limiting point and since all the limiting points are included, it follows that the first derived set M' is equal to the original set M. In the same way the second derived set is equal to the first, that is, $M'' = M'$ and so on for all of the derived sets.

Theorem II. *If an assemblage M is everywhere-dense but not perfect, then its first derived assemblage is perfect.*

Since the set M is everywhere-dense, it contains all its limiting points for as we have seen under definition VIII, page 5, a set that is everywhere-dense is also dense-in-itself. This set also includes other than limiting points since it is not perfect but it contains all its limiting points. The first derived assemblage, however, includes only the limiting points of the original set and since the original set is

everywhere-dense, then will the first derived set contain all its limiting points. Therefore it is perfect.

THEOREM III. *A perfect set need not be in any sub-interval everywhere-dense.*

An example suffices for the proof of this theorem. Such an example is example II, page 6.

Before presenting the next theorem, it will be necessary to give an idea of content. Definitions of content differ in some respects; the definition of G. Cantor will be followed here. The notion of content is very important as the basis of many conceptions of Calculus, especially that of integrability.

Enclose all points of the set in a finite number of intervals and think of the *limit* of the sum of these intervals as they are made as small as possible, as the content of the set of points. If we can make the limit of the sum of the intervals approach zero, then the content of the set is zero. It is very necessary to enclose the limiting points for otherwise we would have an infinite number of intervals.

As an example, let us take the set of positive integers which are less than *91*. If we enclose each of these points in an interval arbitrarily small, we have as the limit of the sum of these intervals as the intervals become smaller and smaller, zero. The content is therefore zero.

As another example, we may take all real numbers equal to or greater than *1* and equal to or less than *2*. These numbers form a set whose content is readily seen to be *1*.

As a matter of interest, we will state the definitions of Peano and Jordan.* Enclose all of the points of the set in a finite number of in-

*Jordan, *Cours d'Analyse*, vol. I, pp. 28, 29.

tervals. Intervals enclosing points of the set are said to enclose only inner points. If E_i is an interval of inner points, then we have the sum of these intervals, ΣE_i, and also we have the sum $\Sigma E_i'$, of the intervals which enclose limiting points not included in the set. Then the limit L of ΣE_i is called the inner content. If the limit of the sum $\Sigma E_i'$ is L', then $L + L'$ is called the outer content. It must be noted that L' may be zero and the set not be closed. This is the case in the following set:

$$1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots\ldots\ldots\ldots, \frac{1}{3^n}.$$

If L' is zero, then the set is a measurable set. The set of irrational numbers is not measurable. If the outer content is zero, then the set is measurable and has the content zero.

Borel* has set up another notion of content. Enclose every point of the set in an arbitrarily small interval. The limit of the sum of these intervals as the intervals grow smaller and smaller is defined as content. Then every enumerable set has the content zero as can readily be seen.

When a set has the content zero, it is said to be discrete.

THEOREM IV. *A perfect assemblage may have the content zero.*

To prove this theorem, it is only necessary to give an example. Example IV, page 9 is such a set for the sum of the intervals cast aside is

$$\Sigma = \frac{1}{3} + \frac{1}{2}[(\frac{2}{3})^2 + (\frac{2}{3})^3 + \ldots\ldots\ldots\ldots]$$

$$= \frac{1}{3} + \frac{1}{2}(\frac{2}{3})^2[1 + \frac{2}{3} + (\frac{2}{3})^2 + \ldots\ldots\ldots\ldots]$$

The geometric series has the sum Σ' given by

---

*Borel, *Leçons sur la Théorie des Fonctions*, p. 46.

$$\Sigma' = \frac{1[1 - (\frac{2}{3})^n]}{\frac{1}{3}} \text{ as } n \text{ becomes infinite}$$

$$= 3.$$

Therefore $$\Sigma = \frac{1}{3} + \frac{1}{2}\cdot\frac{4}{9}\cdot 3$$

$$= \frac{1}{3} + \frac{2}{3} = 1$$

which is the whole interval. Therefore the part left, which is tne content of the set, is zero.

We may also form such a set by the following general method. Take a closed interval (α,β), (*Fig. 4*) of unit length and from the middle of this interval drop a sub-interval of length $\frac{1}{k}$, where k is a finite positive integer. Retain the end-points as points of the set. There will be two intervals left. These intervals will be equal in length. Suppose that the length of each is l. Then from the middle of each of these intervals drop sub-intervals of length $\frac{l}{k}$. Then there will be four intervals left. These will be of equal length, say l_1. From the middle of each of these drop sub-intervals of length $\frac{l_1}{k}$. Continue this process indefinitely, retaining in each case, the end-points of the intervals dropped as points of the set. Each end-point is a limiting point. Also the points α and β are limiting points and must be included. Hence all the limiting points are included in the set and the set is therefore perfect.

The sum of the intervals dropped is

$$\Sigma = \frac{1}{k} + \frac{k-1}{k^2} + \frac{(k-1)^2}{k^3} + \cdots\cdots$$

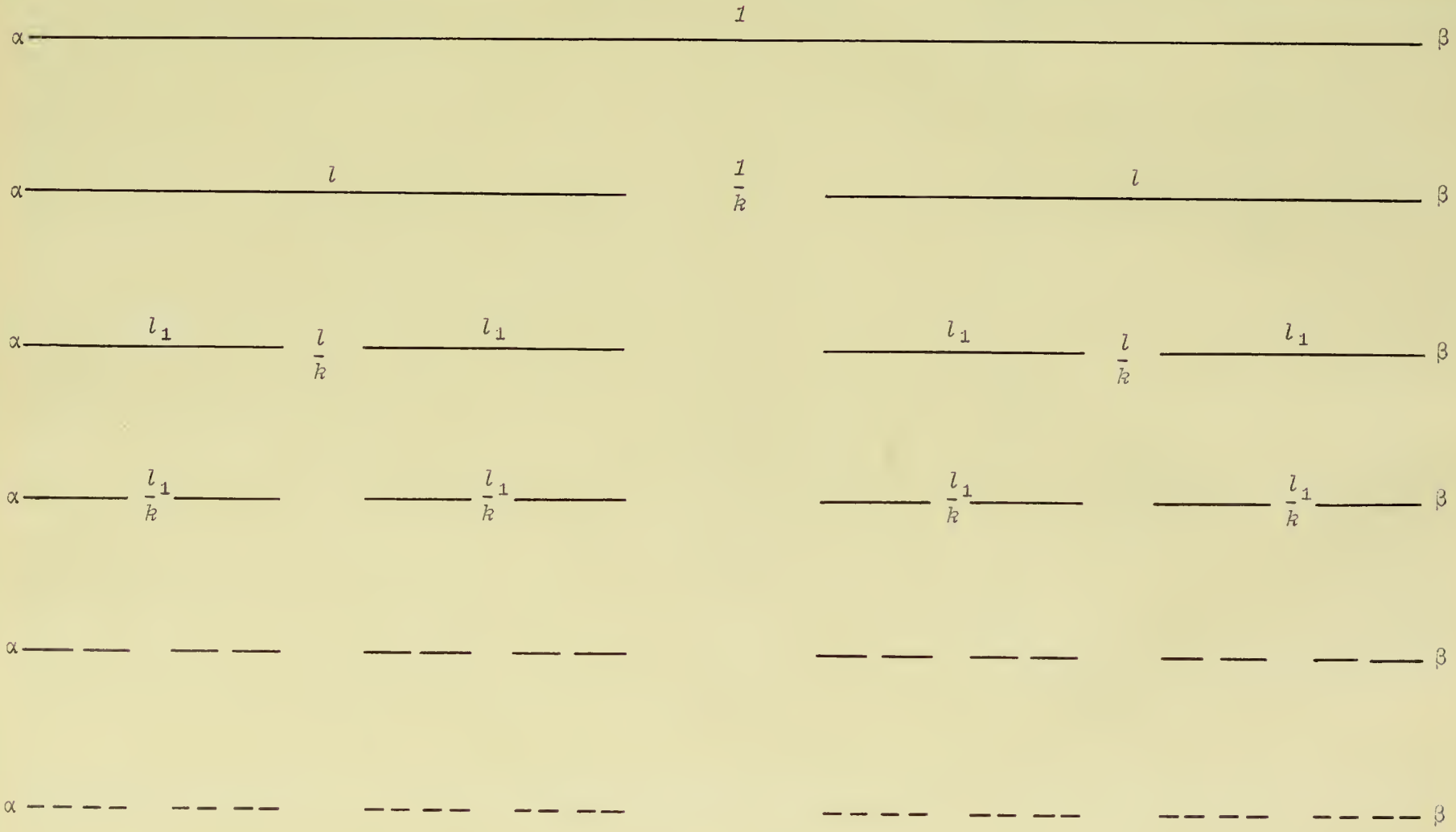

Fig. 4.

($k = 7$).

$$\Sigma = \frac{\frac{1}{k}\left[1 - \left(\frac{k-1}{k}\right)^n\right]}{1 - \frac{k-1}{k}}$$

As n becomes infinite

$$\Sigma = \frac{\frac{1}{k}}{\frac{k - k + 1}{k}} = \frac{\frac{1}{k}}{\frac{1}{k}}$$

$$= 1.$$

The content of the set is, therefore, $1 - 1 = 0$.

CHAPTER II

***

FUNCTIONS AND THE THEORY OF LIMITS.

***

1. CONSTANTS AND VARIABLES. A number which remains unchanged in any particular discussion is called a *constant*.

If an aggregate is given, the numbers in that aggregate may be represented by a *variable* x, where x represents any number or in succession all numbers of the set.* This aggregate is called the *range* or *domain* of the variable x. The domain may be any set of numbers but in this discussion the domain will always be a perfect set. The variable is said to be *continuous* with respect to a given set when it takes all numbers of the set.

2. DEFINITION OF A FUNCTION.° *A single-valued*ˀ *function of a variable* x *is a second variable* y *so related to* x *that whenever a value is assigned to* x *from the* x*-range, a corresponding value of* y *is uniquely determined in the* y*-range.*♀

This correspondence may be made in any way whatsoever. An example is the temperature for the different hours of the day; the temperature is a

---

*Townsend, lectures on "Functions of real variables." For other definitions of a variable see, J. W. A. Young, *Monographs on Topics of Modern Mathematics*, p. 265; Pierpont, *The Theory of Functions of Real Variables*, p. 118; Echols, *Differential and Integral Calculus*, p. 4.

°Due to Dirichlet.

ˀWe shall consider only single-valued functions.

♀Bliss; J. W. A. Young, loc. cit., p. 266.

function of the hours. The range of the dependent variable y is not necessarily exhausted by the correspondence; it may contain elements which do not correspond to any element in the x-range.

Another example of a function is the following. If two points P and Q are joined by a curve C, the area $PQRS$ shown in *Fig. 5* is uniquely determined and depends only on the form of the curve. A functional correspondence is thus set up between the variable C whose range is the totality of curves joining the points P and Q, and the variable A which represents the area.*

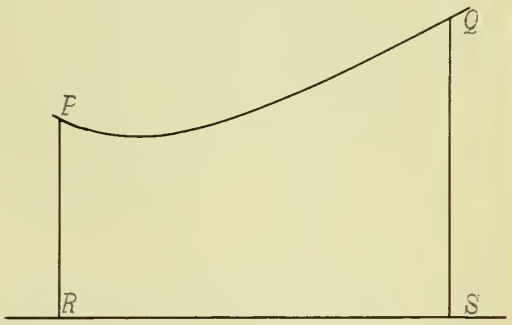

Fig. 5.

Let us take another example of a function.°

For $x > 0$, let $y = 1$.

For $x = 0$, let $y = 0$.

For $x < 0$, let $y = -1$.

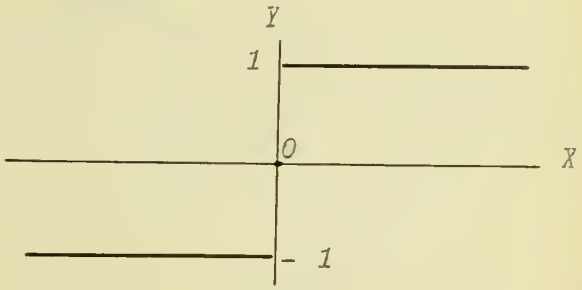

Fig. 6.

This function has the analytic expression

$$y = \frac{2}{\pi} \sum_{n=\infty} \text{arc tan } (nx).$$

It is called *signum x* and is denoted by

$$y = \text{sgn } x.$$

The graph of the function is shown in *Fig. 6*. If we say that sgn $x = -1$, then we indicate that x is negative and if we say that sgn $x = 1$, we in-

---

*This function is given by Bliss; J. W. A. Young, loc. cit., p. 268.

°Pierpont, loc. cit., p. 203.

dicate that x is positive.

A function defined for positive integers is the age y, in days, (at the time the policy was issued), of the holder of policy number x in the Manufacturers Life Insurance Company. Another such function defined only for positive integers is the amount y, in cents, for which Post Office order number x was issued at the Champaign Post Office.

3. Definition of a Limit.* Let P be any perfect set over which the variable x varies, that is, let P be the domain of x. Then by the definition of a perfect set, each point a of this set is a limiting point, that is, x may vary in such a manner that

$$|x - a| < \delta$$

becomes and remains true, when δ is arbitrarily small.

Take any sequence of points of P such that the limit of this sequence is a. Suppose that the sequence is

$$A = a_1, a_2, a_3, a_4, \cdots\cdots\cdots.$$

Then denoting a function of x by $f(x)$, we have the sequence

$$B = f(a_1), f(a_2), f(a_3), \cdots\cdots.$$

Definition I. *If the sequence B has a limit η, finite or infinite, which is a constant, however the sequence A be chosen, then η is the limit of $f(x)$ for $x = a$.*

This is denoted by

$$\mathop{\mathrm{L}}_{x=a} f(x) = \eta$$

Another way of stating this fact is to say that $f(x)$ approaches the limit η as x approaches the limit a. If, however, B has no limit, then we say

---

*This follows the work of Pierpont, loc. cit., pp. 171, 172.

that the limit of $f(x)$ does not exist at $x = a$.

THEOREM I. *Let*

$$A = a_1, a_2, a_3, a_4, \cdots\cdots$$

and

$$B = b_1, b_2, b_3, b_4, \cdots\cdots$$

be any two sequences lying in P, each of which has the limit a. If

$$\underset{a_n=a}{\mathfrak{L}} f(a_n) \neq \underset{b_n=b}{\mathfrak{L}} f(b_n)$$

then the limit of $f(x)$ does not exist at $x = a$.

That this follows at once from the definition of a limit is evident when we notice that B is only another way of choosing the sequence A in the definition.

$$A = a_1, a_2, a_3, a_4, \cdots\cdots$$

may be so chosen that it lies entirely at the right of a or entirely at the left of a. If η exists when A is entirely at the right of a, then it is called a *right-hand limit*. If η exists when A is entirely at the left of a, then it is called a *left-hand limit*. The former case will be denoted by

$$\underset{x=a}{\mathrm{R}\mathfrak{L}} f(x) = \eta$$

and the latter case will be denoted in like manner by

$$\underset{x=a}{\mathrm{L}\mathfrak{L}} f(x) = \eta$$

From the definition of a limit we may state the following theorem:

THEOREM II. *If*

$$\underset{x=a}{\mathfrak{L}} f(x) = \eta$$

where η is finite or infinite, then

$$\underset{x=a}{\mathrm{L}\mathfrak{L}} f(x) = \underset{x=a}{\mathrm{R}\mathfrak{L}} f(x) = \eta$$

and, conversely, if

$$\underset{x=a}{L\mathfrak{C}}\ f(x) = \underset{x=a}{R\mathfrak{C}}\ f(x) = \eta$$

where η *is finite or infinite, then*

$$\underset{x=a}{\mathfrak{C}}\ f(x) = \eta.$$

But, in general, these are not the only limits which the function $f(x)$ has at the point a. We shall now define upper and lower right and left-hand limits at the point a.* Let the domain of x be the perfect set P and let a be a point of this set. Then let $\Upsilon(\delta)$ and $\Lambda(\delta)$ denote the upper and lower limits of the function $f(x)$ for all points of P which are within the interval $(a, a + \delta)$ at the right of a. However, $f(a)$ is not to be considered amongst the functional values of which $\Upsilon(\delta)$ and $\Lambda(\delta)$ are the upper and lower limits.

Now assign a descending sequence of values to δ such that the limit of this sequence is zero. If this sequence be

$$\delta_1, \delta_2, \delta_3, \delta_4, \cdots\cdots\cdots, \delta_n, \cdots\cdots$$

then the sequence

$$\Upsilon(\delta_1), \Upsilon(\delta_2), \Upsilon(\delta_3), \cdots\cdots, \Upsilon(\delta_n), \cdots$$

is such that the members do not increase. They have, therefore, a definite lower limit, which is called the *upper limit of* $f(x)$ *at the right of* a. It may happen that the sequence in $\Upsilon(\delta)$ has no finite lower limit. In this case, the upper limit of $f(x)$ on the right at a is $-\infty$. In any case we shall denote the upper right-hand limit of $f(x)$ at a by $\underset{x=a}{R^{u}\mathfrak{C}}\ f(x)$. Similarly, the sequence

$$\Lambda(\delta_1), \Lambda(\delta_2), \Lambda(\delta_3), \cdots\cdots, \Lambda(\delta_n), \cdots\cdots\cdots$$

is such that its members do not decrease and in general it has an upper limit which is the *lower right-hand limit of* $f(x)$ *at* a. This limit which

---

*Hobson, loc. cit., p. 230.

we shall denote by $\underset{x=a}{R_1\mathfrak{C}}\, f(x)$ may also have infinite values of either sign.

Similarly, upper and lower left-hand limits may be defined. In case a is a limiting point of P on one side only, then the two limits of the function at a on the other side are non-existent.

Thus we have the following definitions:

DEFINITION II. *The upper right-hand limit* $\underset{x=a}{R^u\mathfrak{C}}\, f(x)$ *of the function* $f(x)$ *at the point* a *is the limit of the upper limit of* $f(x)$ *in the open interval* $(a, a+\delta)$ *when* δ *is indefinitely diminished.*

DEFINITION III. *The lower right-hand limit* $\underset{x=a}{R_1\mathfrak{C}}\, f(x)$ *of the function* $f(x)$ *at the point* a *is the limit of the lower limit of* $f(x)$ *in the open interval* $(a, a+\delta)$ *when* δ *is indefinitely diminished.*

DEFINITION IV. *The upper left-hand limit* $\underset{x=a}{L^u\mathfrak{C}}\, f(x)$ *of the function* $f(x)$ *at the point* a *is the limit of the upper limit of* $f(x)$ *in the open interval* $(a, a-\delta)$ *when* δ *is indefinitely diminished.*

DEFINITION V. *The lower left-hand limit* $\underset{x=a}{L_1\mathfrak{C}}\, f(x)$ *of the function* $f(x)$ *at the point* a *is the limit of the lower limit of* $f(x)$ *in the open interval* $(a, a-\delta)$ *when* δ *is indefinitely diminished.*

If

$$\underset{x=a}{R^u\mathfrak{C}}\, f(x) = \underset{x=a}{R_1\mathfrak{C}}\, f(x) = C$$

then it is readily seen that $\underset{x=a}{R\mathfrak{C}}\, f(x)$ exists and is equal to C. Also if

$$\underset{x=a}{L^u\mathfrak{C}}\, f(x) = \underset{x=a}{L_1\mathfrak{C}}\, f(x) = D$$

then $\underset{x=a}{L\mathfrak{C}}\, f(x)$ exists and is equal to D. Since $\Upsilon(\delta)$ and $\Lambda(\delta)$ were chosen independently of $f(a)$, it follows that the limits are independent of $f(a)$.

4. CONTINUITY OF FUNCTIONS. Let $f(x)$ be defined over the perfect

set P. Let a be any point of P. If

$$\underset{x=a}{\mathrm{C}} f(x) = f(a),$$

the function is continuous at a.* Thus we have

DEFINITION I. *If the limit of $f(x)$ at a is the same as the value of $f(x)$ at a, then $f(x)$ is continuous at a.*

We may also express this same condition in the following manner:

DEFINITION II. *$f(x)$ is continuous at a, if for each ε greater than zero, there exists a δ greater than zero, such that*

$$|f(x) - f(a)| < \varepsilon$$

when $|x - a| < \delta$.

But this says only what we find in the following definition:°

DEFINITION III. *The function $f(x)$ is said to be continuous at the point a of the perfect set P over which it is defined, if, corresponding to any arbitrarily chosen positive number ε whatever, a positive number δ dependent on ε can be found, such that $|f(a + \eta) - f(a)| < \varepsilon$, for all positive or negative values of η which are numerically less than δ, and which are such that $a + \eta$ is in the domain P.*

It is readily seen that a function may fulfill the definition for continuity on one side of a point but not on the other side. That is, if

$$\underset{x=a}{\mathrm{RC}} f(x) \neq \underset{x=a}{\mathrm{LC}} f(x)$$

then it may be that either

$$\underset{x=a}{\mathrm{RC}} f(x) = f(a) \text{ in which case the function}$$

---

*Pierpont, loc. cit., p. 208.

°Hobson, loc. cit., p. 221. This definition is due to Cauchy.

has a right-hand continuity at the point a, or

$$\lim_{x=a} f(x) = f(a)$$

in which case the function has a left-hand continuity at the point a.

DEFINITION IV. *A function is continuous in a given interval* (α,β) *if it is continuous at every point in the interval.*

DEFINITION V. *If a function* $f(x)$ *is defined over the perfect set* P, *then* $f(x)$ *is continuous with respect to* P *in a given interval* (α,β) *if it is continuous at every point of* P *which lies in the given interval.*

We may now state the following theorem:

THEOREM I.* *If* (α,β) *be any interval containing points of the perfect set* P, *over which the function* $f(x)$ *is defined, then there is at least one point* x_o *in* (α,β) *such that, in any arbitrarily small neighborhood of* x_o, *the upper limit of the function is the same as the upper limit of the function in the whole interval.*

If M denotes the upper limit of $f(x)$ in the interval (α,β), let the interval be divided into n equal parts. It should be noticed that no restriction is made on M; it may be finite or infinite. In none of these sub-intervals is the upper limit of $f(x)$ greater than M but in at least one of them, say (α_1,β_1), it is equal to M. Divide (α_1,β_1) into n equal parts and in the same way as before, we see that in at least one of these sub-intervals, the upper limit of the function is M. Proceeding in this way indefinitely, we obtain the following sequence of intervals whose lengths converge to zero:

$$(\alpha_1,\beta_1),\ (\alpha_2,\beta_2),\ (\alpha_3,\beta_3),\ \cdots\cdots\cdots,\ (\alpha_n,\beta_n),\ \cdots.$$

Each of these intervals is contained in the one which precedes it and in

---

*Due to Weierstrass. Hobson, loc. cit., p. 224.

each of them the upper limit of the function is M. There is one point x_0 common to these intervals which may be called the limit of the sequence.* This point is such that the upper limit of $f(x)$ in an arbitrarily small neighborhood of x_0 is M.

We may now state a similar theorem in regard to the lower limit of the function $f(x)$.

THEOREM II. *If (α,β) be any interval containing points of the perfect set P, over which $f(x)$ is defined, then there is at least one point x_1 in the interval such that, in any arbitrarily small neighborhood of x_1, the lower limit of the function is the same as the lower limit of the function in the whole interval.*

The proof of this theorem is exactly analogous to the one above and need not be stated.

THEOREM III. *If a function $f(x)$ is defined over the perfect set P, and if $f(x)$ is continuous with respect to P in the interval (α,β), then there exists a finite number G, such that the value of $f(x)$ at any point of P, which lies in (α,β), is not greater than G.*

By theorem I there is at least one point x_0 of P in (α,β) at which the function has the same upper limit as for all points of P in (α,β). But the function is continuous at this point and therefore the limit of the function must be the same as the value of the function at x_0. Also by the definition of continuity, the value of the function at x_0 is finite. Therefore the limit of the function at x_0 is finite and since this is the upper limit of the function in the interval (α,β), the theorem follows at once.

Similarly, we have the following theorem:

---

*For proof of this see Hobson, loc. cit., p. 60.

THEOREM IV. *Given an interval* (α,β) *which contains points of the perfect set* P *over which the function* $f(x)$ *is defined. If* $f(x)$ *is continuous with respect to* P *in the interval* (α,β), *then there exists a number* H *such that the value of* $f(x)$ *is not less than* H *for any point of* P *in* (α,β).

The proof of this theorem will not be stated as it is exactly analogous to the preceding proof.

5. LAWS OF OPERATIONS WITH LIMITS.

THEOREM I. *If a function* $f(x)$ *is defined over a perfect set, and, if* $f(x)$ *has a limit, then the limit of a constant* c *times* $f(x)$ *is equal to* c *times the limit of* $f(x)$.

Suppose that $f(x)$ is defined over the perfect set P and suppose that a is a point of P. Then let*

$$\mathop{\mathrm{L}}_{x=a} f(x) = A.$$

We may then write

$$f(x) = A + \varepsilon(x),$$

where $\varepsilon(x)$ can be made as small as we please by taking x sufficiently near to a. We have, then

$$c\cdot f(x) = c\cdot A + c\cdot\varepsilon(x).$$

Since we can make $\varepsilon(x)$ as small as we please by taking x sufficiently near to a, we can also make $c\cdot\varepsilon(x)$ arbitrarily small in the same way. We have then from the definition of a limit

$$\mathop{\mathrm{L}}_{x=a} c\cdot f(x) = c\cdot A = c\cdot \mathop{\mathrm{L}}_{x=a} f(x),$$

as the theorem requires.

---

*See Townsend and Goodenough, *First Course in Calculus*, p. 20.

It is readily seen that the proof used in this theorem may be specialized to prove the theorem for any of the four fundamental limits. The following corollaries state the theorem for these limits:

COROLLARY I. *If a function $f(x)$ is defined over a perfect set P, and, if $f(x)$ has an upper left-hand limit, then the upper left-hand limit of a constant c times $f(x)$ exists and is equal to c times the upper left-hand limit of $f(x)$.*

COROLLARY II. *If a function $f(x)$ is defined over a perfect set P, and, if $f(x)$ has an upper right-hand limit, then the upper right-hand limit of a constant c times $f(x)$ exists and is equal to c times the upper right-hand limit of $f(x)$.*

COROLLARY III. *If a function $f(x)$ is defined over a perfect set P, and, if $f(x)$ has a lower left-hand limit, then the lower left-hand limit of a constant c times $f(x)$ exists and is equal to c times the lower left-hand limit of $f(x)$.*

COROLLARY IV. *If a function $f(x)$ is defined over a perfect set P, and, if $f(x)$ has a lower right-hand limit, then the lower right-hand limit of a constant c times $f(x)$ exists and is equal to c times the lower right-hand limit of $f(x)$.*

THEOREM II. *If two functions $f(x)$ and $\Phi(x)$ are defined over a perfect set P, and, if at a point $x = a$ of the set P, $f(x)$ has a limit A and $\Phi(x)$ has a limit B, then the sum $[f(x) + \Phi(x)]$ of these functions has a limit C which is equal to the sum $A + B$ of the limits of the functions.*

By hypothesis, we have

$$\mathop{L}_{x=a} f(x) = A$$

and

$$\mathop{L}_{x=a} \Phi(x) = B.$$

We may then write*

$$f(x) = A + \varepsilon_1(x)$$

and

$$\Phi(x) = B + \varepsilon_2(x),$$

where $\varepsilon_1(x)$ and $\varepsilon_2(x)$ are arbitrarily small for values of x sufficiently near to a. Combining these equations, we have

$$f(x) + \Phi(x) = A + B + [\varepsilon_1(x) + \varepsilon_2(x)].$$

Since however, $\varepsilon_1(x) + \varepsilon_2(x)$ can be made as small as we choose, we have from the definition of a limit

$$\lim_{x=a} [f(x) + \Phi(x)] = A + B = \lim_{x=a} f(x) + \lim_{x=a} \Phi(x).$$

That the converse theorem does not necessarily hold is shown by the following example:

Given

$$f(x) = \sin^2 \frac{1}{x}$$

and

$$\Phi(x) = \cos^2 \frac{1}{x}.$$

Then

$$f(x) + \Phi(x) = \sin^2 \frac{1}{x} + \cos^2 \frac{1}{x} = 1.$$

In this case the limit of the sum exists and is 1 at $x = 0$, but the limits of the functions do not exist at that point.

This theorem holds, as did theorem I, for the four fundamental limits. We will not state the theorem for these limits in this form but will extend the theorem to the sum of a finite number of functions as follows:

THEOREM III. *If a finite number of functions*

---

*Townsend and Goodenough, loc. cit., p. 21.

$$f_1(x),\ f_2(x),\ f_3(x),\ \cdots\cdots\cdots\cdots,\ f_n(x)$$

are defined over a perfect set P, and, if at a point $x = a$ *of the set P, each of these functions has a limit,*

$$\lim_{x=a} f_1(x) = A_1,\ \lim_{x=a} f_2(x) = A_2,\ \cdots\cdots\cdots,\ \lim_{x=a} f_n(x) = A_n,$$

then the sum

$$f_1(x) + f_2(x) + f_3(x) + \cdots\cdots\cdots\cdots + f_n(x)$$

of these functions has a limit B which is equal to the sum

$$A_1 + A_2 + A_3 + \cdots\cdots\cdots\cdots + A_n$$

of the limits of the functions.

By hypothesis, we have

$$\lim_{x=a} f_1(x) = A_1$$

$$\lim_{x=a} f_2(x) = A_2$$

$$\cdots\cdots\cdots$$

$$\lim_{x=a} f_n(x) = A_n.$$

These equations may be written in the form

$$f_1(x) = A_1 + \varepsilon_1(x)$$

$$f_2(x) = A_2 + \varepsilon_2(x)$$

$$\cdots\cdots\cdots$$

$$f_n(x) = A_n + \varepsilon_n(x),$$

where

$$\varepsilon_1(x),\ \varepsilon_2(x),\ \varepsilon_3(x),\ \cdots\cdots\cdots\cdots,\ \varepsilon_n(x)$$

are arbitrarily small for values of x sufficiently near to a. Combining these equations, we have

$$f_1(x) + f_2(x) + \cdots + f_n(x) = A_1 + A_2 + \cdots + A_n + \varepsilon_1(x) + \varepsilon_2(x) + \cdots + \varepsilon_n(x).$$

But since n is finite, we may make

$$\varepsilon_1(x) + \varepsilon_2(x) + \varepsilon_3(x) + \cdots + \varepsilon_n(x)$$

arbitrarily small by taking x sufficiently near to a. Then from the definition of a limit, we have

$$\lim_{x=a} [f_1(x) + f_2(x) + \cdots + f_n(x)] = A_1 + A_2 + \cdots + A_n$$

which was to be proved.

Since the proof of this theorem holds equally well for the four fundamental limits, we have the following corollaries:

COROLLARY I. *If a finite number of functions are defined over a perfect set P, and, if at a point $x = a$ of the set P, each of these functions has an upper left-hand limit, then the sum of these functions has an upper left-hand limit which is equal to the sum of the upper left-hand limits of the functions.*

COROLLARY II. *If a finite number of functions are defined over a perfect set P, and, if at a point $x = a$ of the set P, each of these functions has an upper right-hand limit, then the sum of these functions has an upper right-hand limit which is equal to the sum of the upper right-hand limits of the functions.*

COROLLARY III. *If a finite number of functions are defined over a perfect set P, and, if at a point $x = a$ of the set P, each of these func-*

tions has a lower left-hand limit, then the sum of these functions has a lower left-hand limit which is equal to the sum of the lower left-hand limits of the functions.

COROLLARY IV. *If a finite number of functions are defined over a perfect set P, and, if at a point $x = a$ of the set P, each of these functions has a lower right-hand limit, then the sum of these functions has a lower right-hand limit which is equal to the sum of the lower right-hand limits of the functions.*

THEOREM IV. *If two functions $f(x)$ and $\varphi(x)$ are defined over a perfect set P, and, if at a point $x = a$ of the set P, $f(x)$ has a limit A and $\varphi(x)$ has a limit B, then the difference $[f(x) - \varphi(x)]$ of these functions has a limit D which is equal to the difference $A - B$ of the limits of the functions.*

By hypothesis, we have

$$\lim_{x=a} f(x) = A$$

and

$$\lim_{x=a} \varphi(x) = B.$$

We may then write*

$$f(x) = A + \varepsilon_1(x)$$

and

$$\varphi(x) = B + \varepsilon_2(x),$$

where $\varepsilon_1(x)$ and $\varepsilon_2(x)$ are arbitrarily small for values of x sufficiently near to a. Combining these equations, we have

$$f(x) - \varphi(x) = A - B + [\varepsilon_1(x) - \varepsilon_2(x)].$$

Since however, $\varepsilon_1(x) - \varepsilon_2(x)$ can be made as small as we please, we have

*Townsend and Goodenough, loc. cit., p. 21.

from the definition of a limit

$$\mathop{\mathrm{L}}_{x=a} [f(x) - \varphi(x)] = A - B = \mathop{\mathrm{L}}_{x=a} f(x) - \mathop{\mathrm{L}}_{x=a} \varphi(x).$$

The converse theorem does not necessarily hold.

THEOREM V. *If a finite number of functions are defined over a perfect set P, and, if at a point $x = a$ of the set P, each of these functions has a limit, then the limit of the product of the functions is equal to the product of their limits.*

Given

$$\mathop{\mathrm{L}}_{x=a} f_1(x) = A_1$$

$$\mathop{\mathrm{L}}_{x=a} f_2(x) = A_2$$

. . . . . .

$$\mathop{\mathrm{L}}_{x=a} f_n(x) = A_n$$

From this, we have the set of equations

$$f_1(x) = A_1 + \varepsilon_1(x)$$

$$f_2(x) = A_2 + \varepsilon_2(x)$$

. . . . . .

$$f_n(x) = A_n + \varepsilon_n(x)$$

where $\varepsilon_1(x)$, $\varepsilon_2(x)$, ... , $\varepsilon_n(x)$ may be made arbitrarily small by taking x sufficiently near to a. We have by multiplication

$$[f_1(x)][f_2(x)]\cdots\cdots[f_n(x)] = A_1A_2\cdots\cdots A_n + [\text{terms which have at least one } \varepsilon(x) \text{ as a factor}].$$

But the expression in brackets may be made as small as we please by taking

x within an arbitrarily small neighborhood of a. Then from the definition of a limit, we have

$$\mathop{L}_{x=a} [f_1(x)\cdot f_2(x)\cdot\cdots\cdots f_n(x)] = A_1A_2\cdots\cdots A_n$$

which was to be proved.

This leads at once to the following corollaries:

COROLLARY I. *If a finite number of functions are defined over a perfect set P, and, if at a point x = a of the set P, each of these functions has an upper left-hand limit, then the upper left-hand limit of the product of these functions is equal to the product of their upper left-hand limits.*

COROLLARY II. *If a finite number of functions are defined over a perfect set P, and, if at a point x = a of the set P, each of these functions has an upper right-hand limit, then the upper right-hand limit of the product of the functions is equal to the product of their upper right-hand limits.*

COROLLARY III. *If a finite number of functions are defined over a perfect set P, and, if at a point x = a of the set P, each of these functions has a lower left-hand limit, then the lower left-hand limit of the product of the functions is equal to the product of their lower left-hand limits.*

COROLLARY IV. *If a finite number of functions are defined over a perfect set P, and, if at a point x = a of the set P, each of these functions has a lower right-hand limit, then the lower right-hand limit of the product of these functions is equal to the product of their lower*

right-hand limits.

It will be seen that a special case of this theorem arises when a function is raised to a finite integral power. This is the case in which

$$f_1(x) = f_2(x) = \cdots\cdots\cdots\cdots = f_n(x).$$

THEOREM VI. *If each of two functions is defined over a perfect set P, and, if at a point $x = a$ of the set P, each of these functions has a limit, then the limit of the quotient of the functions is equal to the quotient of the limits of the two functions, provided the limit of the denominator is different from zero.*

Given

$$\lim_{x=a} f(x) = A$$

and

$$\lim_{x=a} \varphi(x) = B, \text{ where } B \neq 0.$$

We have, then

$$f(x) = A + \varepsilon_1(x)$$

and

$$\varphi(x) = B + \varepsilon_2(x),$$

where $\varepsilon_1(x)$ and $\varepsilon_2(x)$ are arbitrarily small as before. Dividing, we have

$$\frac{f(x)}{\varphi(x)} = \frac{A + \varepsilon_1(x)}{B + \varepsilon_2(x)} = \frac{A}{B} + \frac{A + \varepsilon_1(x)}{B + \varepsilon_2(x)} - \frac{A}{B} = \frac{A}{B} + \frac{B\varepsilon_1(x) - A\varepsilon_2(x)}{B[B + \varepsilon_2(x)]}.$$

By taking x sufficiently near to a, the last fraction may be made to have an arbitrarily small value for its numerator and to have B^2 for its denominator. Hence the last fraction can be made as small as we please and we may write

$$\lim_{x=a} \frac{f(x)}{\varphi(x)} = \frac{A}{B} = \frac{\lim_{x=a} f(x)}{\lim_{x=a} \varphi(x)}$$

The converse of this theorem does not necessarily hold, however. As

an example, let us take

$$f(x) = \sin x$$

and $$\varphi(x) = \tan x \qquad \text{at } x = \frac{\pi}{2}.$$

Then $$\frac{f(x)}{\varphi(x)} = \cos x$$

which has the limit 0 at $x = \frac{\pi}{2}$. But $\varphi(x)$ has no limit at this point.

This leads to the following corollaries:

COROLLARY I. *If each of two functions is defined over a perfect set P, and, if at a point x = a of the set P, each of these functions has an upper left-hand limit, then the upper left-hand limit of the quotient of these functions is equal to the quotient of the upper left-hand limits of the functions, provided the upper left-hand limit of the denominator is different from zero.*

COROLLARY II. *If each of two functions is defined over a perfect set P, and, if at a point x = a of the set P, each of these functions has an upper right-hand limit, then the upper right-hand limit of the quotient of the functions is equal to the quotient of the upper right-hand limits of the functions, provided the upper right-hand limit of the denominator is different from zero.*

COROLLARY III. *If each of two functions is defined over a perfect set P, and, if at a point x = a of the set P, each of these functions has a lower left-hand limit, then the lower left-hand limit of the quotient of these functions is equal to the quotient of the lower left-hand limits of the functions, provided the lower left-hand limit of the denominator is different from zero.*

COROLLARY IV. *If each of two functions is defined over a perfect set*

P, and, if at a point x = a of the set P, each of the functions has a lower right-hand limit, then the lower right-hand limit of the quotient of these functions is equal to the quotient of the lower right-hand limits of the functions provided the lower right-hand limit of the denominator is different from zero.

THEOREM VII. *If $F(y)$ is a continuous function of y defined over a perfect set P, and if y is any function of x, say $\Phi(x)$, defined over the same set, and, if the limit of y at the point $x = a$ of the set P is b, then*

$$\mathop{\mathrm{L}}_{x=a} F[\Phi(x)] = F[\mathop{\mathrm{L}}_{x=a} \Phi(x)].$$

We have by hypothesis,*

$$\mathop{\mathrm{L}}_{x=a} y = \mathop{\mathrm{L}}_{x=a} \Phi(x) = b, \tag{1}$$

and because of the continuity of $F(y)$, we also have

$$\mathop{\mathrm{L}}_{y=b} F(y) = F(b).$$

We may therefore make $F(y)$ differ by as little as we please from $F(b)$ by properly restricting y to values in the neighborhood of b. From (1), however, $y \doteq b$ as $x \doteq a$; hence for all values of x sufficiently close to a, $F(y) = F[\Phi(x)]$ differs as little as we please from $F(b)$. We may write therefore

$$\mathop{\mathrm{L}}_{x=a} F[\Phi(x)] = F(b) = F[\mathop{\mathrm{L}}_{x=a} \Phi(x)].$$

THEOREM VIII. *If each of three given functions is defined over a perfect set P, and, if one of the given functions lies between the values of the other two functions which approach a common limit, then the given function approaches that same limit.*

---

*Townsend and Goodenough, loc. cit., p. 23.

Suppose that we have*

$$\Phi(x) \leqq f(x) \leqq \Psi(x),$$

where

$$\lim_{x=a} \Phi(x) = \lim_{x=a} \Psi(x) = A.$$

Then we have

$$\Phi(x) = A + \varepsilon_1(x)$$

and

$$\Psi(x) = A + \varepsilon_2(x),$$

where $\varepsilon_1(x)$ and $\varepsilon_2(x)$ can be made as small as we please by taking x sufficiently near to a. Hence for all values of x in the neighborhood of a, we have

$$A + \varepsilon_1(x) \leqq f(x) \leqq A + \varepsilon_2(x),$$

whence

$$\varepsilon_1(x) \leqq f(x) - A \leqq \varepsilon_2(x).$$

But as $\varepsilon_1(x)$ and $\varepsilon_2(x)$ can be made arbitrarily small, it follows that $f(x) - A$ can be made to differ in absolute value as little as we please from zero by the proper selection of x; that is,

$$|f(x) - A| < \sigma$$

where σ is an arbitrarily small positive number. By the definition of a limit, we then have

$$\lim_{x=a} f(x) = A = \lim_{x=a} \Phi(x).$$

6. Continuity of Special Functions. As a result of the properties

---

*Townsend and Goodenough, loc. cit., p. 24.

set forth in the last section, we may now state some properties of functions with regard to their continuity. These will be stated as theorems.

THEOREM I. *If a function $f(x)$ is defined over a perfect set P and is continuous at a point a of the set P, and, if c is any constant, then c times $f(x)$ is also continuous at a.*

We have by hypothesis

$$\lim_{x=a} f(x) = f(a).$$

Also by theorem I, page 25, we know that

$$\lim_{x=a} [c \cdot f(x)] = c \lim_{x=a} f(x)$$

Therefore $$\lim_{x=a} [c \cdot f(x)] = c \cdot f(a)$$

which shows that $c \cdot f(x)$ is continuous at a.

THEOREM II. *If each of two functions, $f(x)$ and $g(x)$ is defined over a perfect set P, and, if each of these functions is continuous at a point a of P, then the function which is the sum or difference of these functions is continuous at a.*

This follows at once from theorem II, page 26, and from theorem IV, page 30, for we have

$$\lim_{x=a} f(x) = f(a)$$

and $$\lim_{x=a} g(x) = g(a)$$

and by the theorems referred to, we have

$$\lim_{x=a} [f(x) \pm g(x)] = \lim_{x=a} f(x) \pm \lim_{x=a} g(x)$$

$$= f(a) \pm g(a)$$

which shows that $f(x) \pm g(x)$ is continuous at the point a.

This theorem may be extended in the case of addition to the following general theorem which, however, will be stated as a corollary to the given theorem.

COROLLARY. *If each of a finite number of functions is defined over a perfect set P, and, if each of these functions is continuous at a point a of the set P, then the sum of these functions is continuous at the point a.*

THEOREM III. *If each of two functions is defined over a perfect set P, and, if each of these functions is continuous at a point a of the set P, then the product of these functions is continuous at a.*

Suppose that the given functions are $f(x)$ and $g(x)$. Then by reason of the continuity of these functions, we have

$$\mathop{\mathrm{L}}_{x=a} f(x) = f(a)$$

and

$$\mathop{\mathrm{L}}_{x=a} g(x) = g(a).$$

Then by theorem V, page 31, we have

$$\mathop{\mathrm{L}}_{x=a} [f(x)\cdot g(x)] = [\mathop{\mathrm{L}}_{x=a} f(x)][\mathop{\mathrm{L}}_{x=a} g(x)]$$

$$= f(a)\cdot g(a)$$

which proves the theorem.

Obviously, this theorem can be generalized as follows:

THEOREM IV. *If each of a finite number of functions is defined over a perfect set P, and, if each of these functions is continuous at a point a of P, then the product of these functions is continuous at a.*

For the special case in which these functions are equal, we have:

COROLLARY. *If a function $f(x)$ is defined over a perfect set P, and*

if $f(x)$ is continuous at a point a of P, then a finite power of $f(x)$ is continuous at the point a.

An important example of this is the following:

Given

$$y = x$$

which is continuous. Then by the corollary just stated,

$y = x^k$ where k is a finite integer, is

also continuous.

THEOREM V. *If each of two functions is defined over a perfect set P, and, if each of these functions is continuous at a point a of P, then the quotient of these functions is continuous at a, provided the value of the denominator at a is not zero.*

Suppose that the given functions are $f(x)$ and $g(x)$. Then since these functions are continuous, we have

$$\lim_{x=a} f(x) = f(a)$$

and

$$\lim_{x=a} g(x) = g(a).$$

Then by theorem VI, page 33,

$$\lim_{x=a} \frac{f(x)}{g(x)} = \frac{\lim_{x=a} f(x)}{\lim_{x=a} g(x)}$$

$$= \frac{f(a)}{g(a)}$$

as was to be proved.

THEOREM VI. *An integral rational function is everywhere continuous over a perfect set P.*

Let

$$y = x^k, \qquad \text{where } k = 1, 2, \cdots\cdots, n.$$

Then by the corollary of theorem IV, page 38, y is continuous at every point of the perfect set P. By theorem I, page 37, $a_{n-k}y$ is continuous and by the corollary of theorem II, page 38, we have

$$a_o x^n + a_1 x^{n-1} + \cdots\cdots\cdots + a_{n-1}x + a_n.$$

continuous, which proves the theorem.

THEOREM VII. *A rational function is continuous at every point of a perfect set P for which the denominator does not vanish.*

Given

$$y = \frac{a_o x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n}{b_o x^n + b_1 x^{n-1} + \cdots + b_{n-1}x + b_n}.$$

By theorem VI, the numerator and denominator of y are continuous at every point of P. Then by theorem V, y is continuous for every point of P at which the denominator is not zero.

THEOREM VIII. *The circular functions* sin x *and* cos x *are continuous at every point of the perfect set P over which they are defined.*

The domain of definition of sin x and cos x embraces all points of the set P.* Also we have for $x = a + \delta$,‡

$$\sin x = \sin (a + \delta) = \sin a \cos \delta + \cos a \sin \delta.$$

Since

$$\lim_{x=a} \sin x = \lim_{\delta=0} \sin (a + \delta),$$

and

---

*Pierpont, loc. cit., p. 126.

‡Pierpont, loc. cit., p. 185.

$$\lim_{\delta=0} \sin \delta = 0$$

and

$$\lim_{\delta=0} \cos \delta = 1,$$

then we have

$$\lim_{x=a} \sin x = \sin a$$

and therefore sin x is continuous at every point of P over which it is defined. Similarly, for $x = a + \delta$,

$$\cos x = \cos (a + \delta) = \cos a \cos \delta - \sin a \sin \delta.$$

Then, since

$$\lim_{x=a} \cos x = \lim_{\delta=0} \cos (a + \delta),$$

we have

$$\lim_{x=a} \cos x = \cos a$$

and cos x is continuous at every point of P.

THEOREM IX. *The circular functions* tan x *and* sec x *are continuous at every point of* P *at which* cos x *is not zero.*

This follows at once from theorems VIII and V.

THEOREM X. *The circular functions* cot x *and* csc x *are continuous at every point of* P *at which* sin x *is not zero.*

This follows at once from theorems VIII and V.

CHAPTER III

***

THE DERIVATIVE AND SOME OF ITS PROPERTIES.

***

1. DEFINITION OF A DERIVATIVE. Suppose that the function

$$y = f(x)$$

is defined over a perfect set P. At a point $x + h$, this becomes

$$y + \Delta y = f(x + h).$$

This leads at once to the following definition:

DEFINITION I.* *If the ratio*

$$\frac{f(x + h) - f(x)}{h}$$

approaches a definite limit as h approaches 0, the derivative of $f(x)$ at the point x is the limit

$$\lim_{h=0} \frac{f(x + h) - f(x)}{h}.$$

It is readily seen that two such limiting processes exist according as h has the positive values or negative values. If h has only positive values, then the given limit defines a right-hand derivative while if h has only negative values, then the given limit defines a left-hand derivative. "It follows from the definition of a derivative that, except in one particular case, it exists only when both these limits exist and are equal.

*Veblen and Lennes, *Introduction to Infinitesimal Analysis*, p. 117.

The exception is the case of the derivative of a function at an end-point of an interval upon which the function is defined."* In this case the derivative is equal to the right- or left-hand derivative as may be the case.

Whether the right- or left-hand derivatives exist or not, "there are always four so-called derived numbers, namely, the upper and lower bounds of indetermination of

$$\frac{f(x + h) - f(x)}{h}$$

as x approaches $x + h$ from the right or from the left."° The following notation will be used:‡

For the upper right-hand derived number: $D^r f(x)$.

For the lower right-hand derived number: $D_r f(x)$.

For the upper left-hand derived number: $D^l f(x)$.

For the lower left-hand derived number: $D_l f(x)$.

The following theorem may now be proved:

THEOREM I. *If a function $f(x)$ is defined and continuous for all points of an interval $(x_o - \varepsilon, x_o + \varepsilon)$ about $x = x_o$, there exists a perfect set P on the right of x_o and such that x_o is a point of the set P, for which the derivative of $f(x)$ at x_o with respect to P exists and has any preassigned value λ between $D^r f(x_o)$ and $D_r f(x_o)$, the upper and lower right-hand derivatives with respect to the continuum, that is,*

$$D_{(P_r)} f(x_o) = \lambda, \qquad D^r f(x_o) \geqq \lambda \geqq D_r f(x_o).$$

---

*Veblen and Lennes, loc. cit., p. 118.

°Veblen and Lennes, loc. cit., p. 119.

‡Henceforth the work will follow very closely that of E. R. Hedrick, *On Derivatives over Assemblages*, TRANS. OF THE AMER. MATH. SOC., vol. 8, (1907), p. 345.

The proof of this theorem follows at once when we notice that

$$\frac{f(x_o + h) - f(x_o)}{h}$$

is continuous for it takes for some value $h < \delta$ any value between its upper and its lower limits in the interval $0 < h < \delta$.

It is evident that a similar theorem could be stated for the derivatives at the left of x_o. We may also make the following statements which will be put in the form of theorems.*

THEOREM II. *If the derivative of $f(x)$ exists at a point $x = x_o$ with respect to a finite number of perfect sets P_1, P_2, P_3, ·····, P_n, and has the same value for each of them, the derivative of $f(x)$ exists for the total set $\sum_1^n P$ formed by combining the given sets and has the same value.*

THEOREM III. *A necessary and sufficient condition that $f(x)$ have a derivative at $x = x_o$ with respect to any perfect set P is that the derivative of $f(x)$ should exist and have the same value for every possible sequence in P.*

2. THE LAW OF THE MEAN. A very general theorem may be stated as follows:

THEOREM I. *Let a function $f(x)$ be defined over a perfect set P. If $f(x)$ has a maximum with respect to P at a point $x = x_o$ of P, then*

$$D^l_{(E)}f(x_o) \geqq D_{l(E)}f(x_o) \geqq 0 \geqq D^r_{(E)}f(x_o) \geqq D_{r(E)}f(x_o)$$

where E is P or any sub-assemblage of P for which x_o is a limiting point. (It does not follow that E is perfect).

This follows at once since for sufficiently small values of $|\delta|$

*E. R. Hedrick, loc. cit., p. 346, 347.

$$f(x_o - h) < f(x_o)$$

and therefore

$$\frac{f(x_o - h) - f(x_o)}{-h}$$

is positive for all values of h such that $0 < h \leq |\delta|$. Also for values of $|\delta|$ sufficiently small

$$f(x_o + h) < f(x_o)$$

and therefore

$$\frac{f(x_o + h) - f(x_o)}{h}$$

is negative for all values of h such that $0 < h \leq |\delta|$.

COROLLARY I. *If a given function $f(x)$ is defined over a perfect set P and if there exists at each point x of P, (except, of course, at the right-hand bound), an assemblage R_x^* having points near to x on the right and such that the derivative $D_{(R_x)}f(x)$ with respect to R_x exists and is positive, then $f(x)$ is at a maximum, if at all, at a right-hand boundary.*

This follows at once from the theorem just proved for if $f(x)$ had a maximum at a point which is not a right-hand boundary, then $D_{(R_x)}f(x)$ would be less than or equal to zero. But this is contrary to hypothesis. Therefore the corollary is proved.

A similar corollary could be stated for the left-hand boundary.

COROLLARY II. (Rolle's Theorem). *If a function $f(x)$ is defined over a perfect set P and if $f(x)$ has a maximum at $x = x_o$ where x_o is not a boundary point, then*

$$L \geq 0 \geq l$$

where L and l are respectively the upper and lower limits of $D_{(R_x)}f(x)$, [or

*This assemblage may be the part of P which lies at the right of x.

$D_{(L_x)}f(x)]$, *where* R_x *and* L_x *are sets chosen arbitrarily at the right and left of* x, *for every point* x *which is not a boundary (right or left).*

This is true for if

$$l = D_{(R_x)}f(x) > 0,$$

then $f(x)$ has a maximum at a right-hand boundary only.

Let us now consider the function

$$\Phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

defined over a perfect set P of which a and b are points. Then

$$\Phi(a) = \Phi(b) = 0.$$

COROLLARY III. (Law of the Mean). *If the function* $f(x)$ *is defined and continuous over a perfect set* P, *then*

$$L \geqq \frac{f(b) - f(a)}{b - a} \geqq l$$

where L *and* l *are the upper and lower limits respectively of the values of* $D_{(R_x)}f(x)$ *where* R_x *is an assemblage chosen arbitrarily, as above, for each value of* x.

3. DETERMINATION OF A FUNCTION BY MEANS OF DERIVATIVES.

THEOREM I. *If a function* $f(x)$ *is defined and continuous over a perfect set* P *and if* $D_{(R_x)}f(x)$ *is equal to zero for some right-hand assemblage* R_x *at each point* x *of the set* P *within an interval* (α,β) *where* α *is any left-hand boundary and* β *is the first boundary at the right of* α,*

*It must be noted that such an interval is very general and is, in the case of the continuum, any finite interval as large as we choose.

then $f(x)$ has a constant value for all values of x in the given interval.

For, let*

$$\varphi(x) = f(x) + \varepsilon(x - \alpha),$$ where ε is an arbitrarily small positive number, and

$$\psi(x) = f(x) - \varepsilon(x - \alpha).$$

Then

$$D_{(R_x)}\varphi(x) = \varepsilon > 0$$

and

$$D_{(R_x)}\psi(x) = -\varepsilon < 0.$$

Hence $\varphi(x)$ is at a maximum at β and $\psi(x)$ is at a minimum at β. Then

$$\varphi(\beta) \geqq \varphi(x) \geqq f(x) \geqq \psi(x) \geqq \psi(\beta)$$

for all values of x such that $\alpha \leqq x < \beta$. From this, we have

$$f(\beta) + \varepsilon(\beta - \alpha) = f(x) = f(\beta) - \varepsilon(\beta - \alpha).$$

It follows now that $f(x)$ differs from $f(\beta)$ by at most only the arbitrary, small positive number $\varepsilon(\beta - \alpha)$. Therefore

$$f(x) = f(\beta).$$

THEOREM II. *If two functions $f(x)$ and $\varphi(x)$ are defined and continuous over a perfect set P and if $D_{(P)}f(x)$ and $D_{(P)}\varphi(x)$ both exist and are equal in any interval (α,β) defined as above, then the two functions differ at most by only a constant within that interval.*

For, let us define a function

---

*Hedrick, loc. cit., p. 349. Also Dini, *Theorie der Funktionen einer veränderlichen reellen Grösse*, p. 113.

$$\psi(x) = f(x) - \varphi(x).$$

Then since $D_{(P)}f(x)$ and $D_{(P)}\varphi(x)$ both exist, we have

$$D_{(P)}\psi(x) = D_{(P)}f(x) - D_{(P)}\varphi(x) = 0$$

for $\alpha \leq x \leq \beta$. Then by theorem I, $\psi(x)$ is a constant. Therefore if $\psi(x) = C$, then

$$C = f(x) - \varphi(x)$$

or $f(x) = \varphi(x) + C$, where C may take the value zero or may be negative.

4. Relationship of Derivatives. We may add a theorem which is interesting because of the manner in which it connects any perfect set with the continuum.

Theorem. *If $f(x)$ is defined and continuous over a perfect set P and if the right-hand derivative $D_{(P)r}f(x)$ of $f(x)$ with respect to P exists and is continuous at each point x in an interval $\alpha \leq x < \beta$ as defined above, that is, if*

$$D_{(P)r}f(x) = \varphi(x)$$

where $\varphi(x)$ is continuous, then the derivative of $f(x)$ with respect to the continuum exists and is equal to $\varphi(x)$.

For* if $D_{(P)r}f(x) = \varphi(x)$ exists and is continuous, then the integral

$$\psi(x) = \int_a^x \varphi(x)dx$$

exists and is continuous and has a derivative with respect to the continuum

*This proof is entirely due to Hedrick, loc. cit., p. 860.

which is

$$\frac{d\psi(x)}{dx} = \varphi(x) \qquad \text{for } \alpha < x < \beta$$

and

$$\left.\frac{d\psi(x)}{dx}\right|_{(r)} = \varphi(a) \qquad \text{for } x = a,$$

where (r) indicates the right-hand derivative with respect to the continuum. Hence $f(x)$ and $\psi(x)$ are two continuous functions for which

$$D_{(p)}{}^{r}f(x) = D_{(p)}{}^{r}\psi(x) = \frac{d\psi(x)}{dx} \qquad \text{for } \alpha \leqq x < \beta.$$

Then

$f(x) = \psi(x) + C$, where C is a constant and

hence $f(x)$ has a derivative with respect to the continuum.

UNIVERSITY OF ILLINOIS-URBANA
3 0112 086859540